

ON THE STABILITY OF THE FIRST ORDER LINEAR RECURRENCE IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT. Suppose that \mathcal{X} is a sequentially complete Hausdorff locally convex space over a scalar field \mathbb{K} , V is a bounded subset of \mathcal{X} , $(a_n)_{n \geq 0}$ is a sequence in $\mathbb{K} \setminus \{0\}$ with the property $\liminf_{n \rightarrow \infty} |a_n| > 1$ and $(b_n)_{n \geq 0}$ is a sequence in \mathcal{X} . We show that for every sequence $(x_n)_{n \geq 0}$ in \mathcal{X} satisfying

$$x_{n+1} - a_n x_n - b_n \in V \quad (n \geq 0)$$

there exists a unique sequence $(y_n)_{n \geq 0}$ satisfying the recurrence $y_{n+1} = a_n y_n + b_n$ ($n \geq 0$) and for every q with $1 < q < \liminf_{n \rightarrow \infty} |a_n|$, there exists $n_0 \in \mathbb{N}$ such that

$$x_n - y_n \in \frac{1}{q-1} \overline{\text{conv}(V^b)} \quad (n \geq n_0).$$

1. INTRODUCTION

The stability problem of functional equations was originally raised by Ulam [19] in 1940 on a talk at Wisconsin University. The problem posed by Ulam was the following: “Under what conditions does there exist an additive mapping near an approximately additive mapping?” The first answer to the question was given by Hyers in the case of Banach spaces [7]. After Hyers’ result many papers dedicated to this topic extending Ulam’s problem to other functional equations and generalizing Hyers’ result in various directions were published (see e.g. [4, 6, 8, 9, 10, 11, 16]). As mentioned in [1] there are much less results on the stability for functional equations in single variable than in several variables. A particular case of equation in single variable is the linear recurrence (difference equation)

$$x_{n+1} = a_n x_n + b_n. \tag{1.1}$$

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The results on stability of recurrences play an important role in the theory of dynamical systems and computer science in connection to the notions of shadowing and controlled chaos (see e.g. [12, 13, 18]). The first result on Hyers–Ulam stability of the linear recurrence (1.1) was given by Popa [14] in the case of Banach spaces as follows.

Theorem 1.1. [14] *Let \mathcal{X} be a Banach space over the field \mathbb{K} (\mathbb{R} or \mathbb{C}), $(\varepsilon_n)_{n \geq 0}$ a sequence of positive numbers, $(a_n)_{n \geq 0}$ a sequence in $\mathbb{K} \setminus \{0\}$ with the property*

$$\limsup_{n \rightarrow \infty} \frac{\varepsilon_n}{\varepsilon_{n-1}|a_n|} < 1 \quad \text{or} \quad \liminf_{n \rightarrow \infty} \frac{\varepsilon_n}{\varepsilon_{n-1}|a_n|} > 1 \quad (1.2)$$

and $(b_n)_{n \geq 0}$ a sequence in \mathcal{X} .

Then there exists a constant $L > 0$ such that for every sequence $(x_n)_{n \geq 0}$ in \mathcal{X} satisfying the relation

$$\|x_{n+1} - a_n x_n - b_n\| \leq \varepsilon_n \quad (n \geq 0) \quad (1.3)$$

there exists a sequence $(y_n)_{n \geq 0}$ given by the linear recurrence

$$y_{n+1} = a_n y_n + b_n \quad (n \geq 0)$$

with the property

$$\|x_n - y_n\| \leq L \varepsilon_{n-1}. \quad (1.4)$$

for some $n_0 \geq 0$ and all $n \geq n_0$.

The above result was extended later by Brzdek, Popa and Xu to the linear recurrences of higher order with constant coefficients [15] and to nonlinear recurrences [2] (see also [3]).

The goal of this paper is to extend Theorem 1.1 to the stability of the linear recurrence (1.1) in topological vector spaces (see also [5]).

If \mathcal{X} is a topological vector space over the field \mathbb{K} , $(a_n)_{n \geq 0}$ a sequence in \mathbb{K} , $(b_n)_{n \geq 0}$ a sequence in \mathcal{X} and V a bounded subset of \mathcal{X} , then the recurrence (1.1) is said to be stable in the spirit of Hyers–Ulam whenever there exists a bounded subset W of \mathcal{X} such that for any sequence $(x_n)_{n \geq 0}$ in \mathcal{X} satisfying

$$x_{n+1} - a_n x_n - b_n \in V \quad (n \geq 0) \quad (1.5)$$

there exists a sequence $(y_n)_{n \geq 0}$ satisfying the linear recurrence

$$y_{n+1} = a_n y_n + b_n, \quad (n \geq 0) \quad (1.6)$$

and

$$x_n - y_n \in W \quad (1.7)$$

for some $n_0 \geq 0$ and all $n \geq n_0$. For a subset A of a topological vector space \mathcal{X} over the field \mathbb{K} we denote the closedness, the balanced hull and the convex hull of A by \overline{A} , A^b and $\text{conv}A$, respectively. Recall that for $\lambda, \mu \in \mathbb{K}$ the following relation holds $(\lambda + \mu)A \subseteq \lambda A + \mu A$. If \mathbb{K} is one of the fields \mathbb{R} or \mathbb{C} , A is a convex set and λ, μ are positive numbers, then $(\lambda + \mu)A = \lambda A + \mu A$. The reader is referred to [17] for undefined notation and terminology.

2. MAIN RESULTS

We give, for the beginning, two auxiliary lemmas which will be used to obtain the main result of this paper. Throughout this section $\mathbb{N}_0, \mathbb{R}, \mathbb{C}$ stand as usual for the set of all nonnegative integers, real numbers and complex numbers, respectively. By \mathbb{K} we denote one of the fields \mathbb{R} or \mathbb{C} .

Lemma 2.1. *Let \mathcal{X} be a sequentially complete Hausdorff locally convex space over \mathbb{K} . Suppose that $(a_n)_{n \geq 0}$ is a sequence in $\mathbb{K} \setminus \{0\}$ such that the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent and let $(v_n)_{n \geq 0}$ be a bounded sequence in \mathcal{X} . Then the series*

$$\sum_{n=0}^{\infty} a_n v_n$$

is convergent in \mathcal{X} .

Proof. Let V be an arbitrary neighborhood of 0 and set

$$\sigma_n = \sum_{k=0}^n a_k v_k, \quad s_n = \sum_{k=0}^n |a_k| \quad (n \geq 0).$$

We have to prove that there exists $n_V \in \mathbb{N}_0$ such that

$$\sigma_{n+p} - \sigma_n \in V$$

for every $n \geq n_V$ and every $p \in \mathbb{N}_0$.

Let U be a convex and balanced neighborhood of 0 such that $U \subseteq V$.

Since $A = \{v_n \mid n \in \mathbb{N}\}$ is a bounded set, it follows that there exists $\alpha > 0$ such that $A \subseteq \alpha U$.

From the convergence of $\sum_{n=0}^{\infty} |a_n|$ follows that there exists $n_0 \in \mathbb{N}$ such that

$$s_{n+p} - s_n < \frac{1}{\alpha} \quad (n \geq n_0, p \in \mathbb{N}).$$

We have

$$\begin{aligned} \sigma_{n+p} - \sigma_n &= \sum_{k=n+1}^{n+p} a_k v_k \\ &= (s_{n+p} - s_n) \sum_{k=n+1}^{n+p} \frac{a_k}{s_{n+p} - s_n} \cdot v_k \\ &\in \alpha(s_{n+p} - s_n) \sum_{k=n+1}^{n+p} \frac{a_k}{s_{n+p} - s_n} U. \end{aligned}$$

Since U is a balanced set the following relation holds

$$\frac{a_k}{s_{n+p} - s_n} U = \frac{|a_k|}{s_{n+p} - s_n} U \quad (n+1 \leq k \leq n+p)$$

hence

$$\sigma_{n+p} - \sigma_n \in \alpha(s_{n+p} - s_n) \sum_{k=n+1}^{n+p} \frac{|a_k|}{s_{n+p} - s_n} U.$$

The convexity of U leads to

$$\sum_{k=n+1}^{n+p} \frac{|a_k|}{s_{n+p} - s_n} U = \left(\sum_{k=n+1}^{n+p} \frac{|a_k|}{s_{n+p} - s_n} \right) U = U$$

therefore

$$\sigma_{n+p} - \sigma_n \in \underbrace{\alpha(s_{n+p} - s_n)}_{<1} U \subseteq U \subseteq V$$

for every $n \geq n_0$ and every $p \in \mathbb{N}$. □

Lemma 2.2. *Let \mathcal{X} be a vector space over \mathbb{K} , $(a_n)_{n \geq 0}$ be a sequence in \mathbb{K} , $(b_n)_{n \geq 0}$ a sequence in \mathcal{X} and $(x_n)_{n \geq 0}$ a sequence in \mathcal{X} satisfying the recurrence (1.1). Then*

$$x_n = a_0 a_1 \dots a_{n-1} x_0 + \sum_{k=1}^{n-1} a_k \dots a_{n-1} b_{k-1} + b_{n-1} \quad (n \geq 2).$$

Proof. Induction on n . □

The first stability result for recurrence (1.1) is given in the next theorem.

Theorem 2.3. *Suppose that \mathcal{X} is a sequentially complete Hausdorff locally convex space over \mathbb{K} , V is a bounded subset of \mathcal{X} , $(a_n)_{n \geq 0}$ is a sequence in $\mathbb{K} \setminus \{0\}$ with the property $\liminf_{n \rightarrow \infty} |a_n| > 1$ and $(b_n)_{n \geq 0}$ is a sequence in \mathcal{X} . Then for every sequence $(x_n)_{n \geq 0}$ in \mathcal{X} satisfying*

$$x_{n+1} - a_n x_n - b_n \in V \quad (n \geq 0) \quad (2.1)$$

there exists a unique sequence $(y_n)_{n \geq 0}$ satisfying the recurrence

$$y_{n+1} = a_n y_n + b_n \quad (n \geq 0) \quad (2.2)$$

and for every q with $1 < q < \liminf_{n \rightarrow \infty} |a_n|$, there exists $n_0 \in \mathbb{N}$ such that

$$x_n - y_n \in \frac{1}{q-1} \overline{\text{conv}(V^b)} \quad (n \geq n_0). \quad (2.3)$$

Proof. Existence. Let $(x_n)_{n \geq 0}$ be a sequence in \mathcal{X} satisfying (2.1) and define the sequence $(c_n)_{n \geq 0}$ by

$$c_n = x_{n+1} - a_n x_n - b_n \quad (n \geq 0). \quad (2.4)$$

Taking account of

$$\limsup_{n \rightarrow \infty} \frac{\frac{1}{|a_0 \cdots a_{n+1}|}}{\frac{1}{|a_0 \cdots a_n|}} = \limsup_{n \rightarrow \infty} \frac{1}{|a_{n+1}|} < 1$$

it follows that the series

$$\sum_{n=0}^{\infty} \frac{1}{|a_0 \cdots a_n|} \quad (2.5)$$

is convergent in view of D'Alembert ratio test.

The boundedness of $(c_n)_{n \geq 0}$ and the convergence of the series (2.5) implies the convergence of the series $\sum_{n=0}^{\infty} \frac{c_n}{a_0 a_1 \cdots a_n}$ in view of Lemma 2.1. Put

$$\sum_{n=0}^{\infty} \frac{c_n}{a_0 a_1 \cdots a_n} = s \quad (s \in \mathcal{X})$$

and define the sequence $(y_n)_{n \geq 0}$, by the recurrence (2.2) with

$$y_0 = x_0 + s.$$

It follows from Lemma 2.2 that

$$\begin{aligned}
x_n - y_n &= - \prod_{k=0}^{n-1} a_k s + \sum_{k=1}^{n-1} a_k \dots a_{n-1} c_{k-1} + c_{n-1} \\
&= a_0 a_1 \dots a_{n-1} \left(-s + \sum_{k=1}^n \frac{c_{k-1}}{a_0 \dots a_{k-1}} \right) \\
&= a_0 \dots a_{n-1} \sum_{k=0}^{\infty} \frac{c_{n+k}}{a_0 a_1 \dots a_{n+k}} \\
&= \sum_{k=0}^{\infty} \frac{c_{n+k}}{a_n a_{n+1} \dots a_{n+k}}. \tag{2.6}
\end{aligned}$$

Let q be a real number such that

$$1 < q < \liminf_{n \rightarrow \infty} |a_n|.$$

Then, there exists $n_0 \in \mathbb{N}$ such that $|a_n| \geq q$ for every $n \geq n_0$.

The following relations hold

$$\begin{aligned}
\frac{c_{n+k}}{a_n \dots a_{n+k}} &\in \frac{1}{a_n \dots a_{n+k}} V \subseteq \frac{1}{a_n \dots a_{n+k}} V^b = \frac{1}{|a_n \dots a_{n+k}|} V^b \\
&\subseteq \frac{1}{q^{k+1}} V^b \subseteq \frac{1}{q^{k+1}} \text{conv}(V^b) \quad (n, k \in \mathbb{N}_0, n \geq n_0). \tag{2.7}
\end{aligned}$$

From (2.6) and (2.7) we have

$$\begin{aligned}
\sum_{k=0}^p \frac{c_{n+k}}{a_n \dots a_{n+k}} &\in \sum_{k=0}^p \frac{1}{q^{k+1}} \text{conv}(V^b) \\
&= \left(\sum_{k=0}^p \frac{1}{q^{k+1}} \right) \text{conv}(V^b) \quad (n \geq n_0). \tag{2.8}
\end{aligned}$$

By letting $p \rightarrow \infty$ in (2.8) we get

$$x_n - y_n \in \frac{1}{q-1} \overline{\text{conv}(V^b)} \quad (n \geq n_0).$$

The existence is proved.

Uniqueness. Let $(x_n)_{n \geq 0}$ be a sequence satisfying (2.1) and suppose that there exists a sequence $(y_n)_{n \geq 0}$ satisfying (2.2) and (2.3) with $y_0 \neq x_0 + s$. We

have

$$\begin{aligned} x_n - y_n &= \prod_{k=0}^{n-1} a_k (x_0 - y_0) + \sum_{k=1}^{n-1} a_k \dots a_{n-1} c_{k-1} + c_{n-1} \\ &= \prod_{k=0}^{n-1} a_k \left(x_0 - y_0 + \sum_{k=1}^n \frac{c_{k-1}}{a_0 a_1 \dots a_{k-1}} \right). \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \left(x_0 - y_0 + \sum_{k=1}^n \frac{c_{k-1}}{a_0 a_1 \dots a_{k-1}} \right) = x_0 - y_0 + s \neq 0$$

and

$$\lim_{n \rightarrow \infty} \left| \prod_{k=0}^{n-1} a_k \right| = \infty,$$

in view of the convergence of $\sum_{n=0}^{\infty} \frac{1}{|a_0 \dots a_n|}$, it follows that $(x_n - y_n)_{n \geq 0}$ is an unbounded sequence, a contradiction to (2.3). \square

A similar result holds for the linear recurrence with constant coefficients.

Theorem 2.4. *Let \mathcal{X} be a sequentially complete Hausdorff locally convex space over \mathbb{K} , V a bounded subset of \mathcal{X} , $a \in \mathbb{K}$, $|a| > 1$, and $(b_n)_{n \geq 0}$ a sequence in \mathcal{X} . Then for every sequence $(x_n)_{n \geq 0}$ in \mathcal{X} satisfying*

$$x_{n+1} - ax_n - b_n \in V \quad (n \geq 0) \quad (2.9)$$

there exists a unique sequence $(y_n)_{n \geq 0}$ fulfilling the recurrence

$$y_{n+1} = ay_n + b_n \quad (n \geq 0) \quad (2.10)$$

such that

$$x_n - y_n \in \frac{1}{|a| - 1} \cdot \overline{\text{conv}(V^b)} \quad (n \geq 0).$$

Proof. Denoting $c_n := x_{n+1} - ax_n - b_n$ ($n \geq 0$) it follows that the series $\sum_{n=0}^{\infty} \frac{c_n}{a^{n+1}}$ is convergent in view of Lemma 2.1. Put

$$\sum_{n=0}^{\infty} \frac{c_n}{a^{n+1}} = s \quad (s \in \mathcal{X})$$

and define $(y_n)_{n \geq 0}$ by the recurrence (2.10) with $y_0 = x_0 + s$. It follows, as in the proof of Theorem 2.3, that

$$x_n - y_n = \sum_{k=0}^{\infty} \frac{c_{n+k}}{a^{k+1}} \quad (n \geq 0)$$

and

$$\frac{c_{n+k}}{a^{k+1}} \in \frac{1}{|a|^{k+1}} \cdot V^b \subseteq \frac{1}{|a|^{k+1}} \cdot \text{conv}(V^b).$$

Then

$$\sum_{k=0}^p \frac{c_{n+k}}{a^{k+1}} \in \left(\sum_{k=0}^p \frac{1}{|a|^{k+1}} \right) \text{conv}(V^b). \quad (2.11)$$

Now by letting $p \rightarrow \infty$ in (2.11) we get (2.10).

The uniqueness follows analogously to Theorem 2.3. \square

Corollary 2.5. *Suppose that \mathcal{X} is a Banach space over \mathbb{K} , $\varepsilon > 0$, $|a| > 1$ and $(b_n)_{n \geq 0}$ is a sequence in \mathcal{X} . Then for every sequence $(x_n)_{n \geq 0}$ in \mathcal{X} satisfying*

$$\|x_{n+1} - ax_n - b_n\| \leq \varepsilon \quad (n \geq 0)$$

there exists a unique sequence $(y_n)_{n \geq 0}$ satisfying the recurrence

$$y_{n+1} = ay_n + b_n \quad (n \geq 0)$$

such that

$$\|x_n - y_n\| \leq \frac{\varepsilon}{|a| - 1} \quad (n \geq 0).$$

Proof. Use Theorem 2.4 with $a_n = a$ and take V to be the closed ball of center 0 with radius ε . \square

Remark 2.6. If $\liminf_{n \rightarrow \infty} |a_n| \leq 1$, then the conclusion of Theorem 2.3 may be not true.

To see this, set $\mathcal{X} = \mathbb{K} = \mathbb{R}$, take $1 \leq r < 2$, let V be the interval $(-r, r)$, $a_n = r$, $b_n = 0$ ($n \geq 0$) and consider the sequence $(x_n)_{n \geq 0}$ given by $x_{n+1} - rx_n = 1$ ($n \geq 0$), $x_0 = 0$. Then $x_n = \sum_{j=1}^{n-1} r^j$. One can observe that for any sequence $(y_n)_{n \geq 0}$ satisfying the recurrence $y_{n+1} = ry_n$ we have

$$\sup_{n \rightarrow \infty} |x_n - y_n| = \infty.$$

In fact,

$$\sup_{n \rightarrow \infty} |x_n - y_n| = \sup \left(\left\{ \left| \sum_{j=0}^{n-1} r^j - r^n y_0 \right| : n = 1, 2, \dots \right\} \cup \{|y_0|\} \right).$$

If $y_0 \leq 0$, then $\sup_{n \rightarrow \infty} |x_n - y_n| = \infty$. Let us assume $y_0 > 0$. There exist positive integers k_0 and n_0 such that $r^{k_0} \geq y_0$ and $\sum_{j=0}^{n-1} r^j \geq r^n$ for all $n \geq n_0$.

Then

$$\sum_{j=k_0}^{n-1+k_0} r^j - r^n y_0 \leq r^{k_0} \left(\sum_{j=0}^{n-1} r^j - r^n \right) = (2-r)r^{n+k_0} - r^{k_0}$$

for all $n \geq \max\{k_0, n_0\}$. Therefore $\left| \sum_{j=0}^{n-1} r^j - r^n y_0 \right| \rightarrow \infty$ as $n \rightarrow \infty$. Hence $\sup_{n \rightarrow \infty} |x_n - y_n| = \infty$.

Theorem 2.7. *Let \mathcal{X} be a Hausdorff topological vector space over the field \mathbb{K} , V a bounded subset of \mathcal{X} , $(a_n)_{n \geq 0}$ a sequence in \mathbb{K} with $\limsup_{n \rightarrow \infty} |a_n| < 1$ and $(b_n)_{n \geq 0}$ a sequence in \mathcal{X} . Then there exists a positive number M such that for every sequence $(x_n)_{n \geq 0}$ in \mathcal{X} satisfying*

$$x_{n+1} - a_n x_n - b_n \in V \quad (n \geq 0)$$

there exists a sequence $(y_n)_{n \geq 0}$ satisfying the linear recurrence

$$y_{n+1} = a_n y_n + b_n \quad (n \geq 0) \tag{2.12}$$

such that

$$x_n - y_n \in M \cdot \text{conv}(V^b) \quad (n \geq n_0).$$

Proof. Let $(c_n)_{n \geq 0}$ be defined by (2.4), as in the proof of Theorem 2.3 and $(y_n)_{n \geq 0}$ be given by the recurrence (2.12) with $y_0 = x_0$. Then

$$x_n - y_n = \sum_{k=1}^{n-1} a_k \dots a_{n-1} c_{k-1} + c_{n-1} \quad (n \geq 2).$$

Choose $q \in \mathbb{R}$ such that

$$\limsup_{n \rightarrow \infty} |a_n| < q < 1.$$

Then there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \leq q \quad (n \geq n_0).$$

For $k \in \mathbb{N}$, $k \geq n_0$ we have

$$|a_k \dots a_{n-1}| \leq q^{n-k}$$

and for $k < n_0$

$$\begin{aligned} |a_k \dots a_{n-1}| &= |a_k \dots a_{n_0-1}| \cdot |a_{n_0} \dots a_{n-1}| \\ &\leq q^{n-n_0} |a_k \dots a_{n_0-1}| \\ &\leq |a_k \dots a_{n_0-1}|. \end{aligned}$$

Then, the same argument as in the proof of Theorem 2.3 follows

$$\begin{aligned} x_n - y_n &= \sum_{k=1}^{n_0-1} a_k \dots a_{n-1} c_{k-1} + \sum_{k=n_0}^{n-1} a_k \dots a_{n-1} c_{k-1} + c_{n-1} \\ &\in \sum_{k=1}^{n_0-1} |a_k \dots a_{n_0-1}| \cdot V^b + \sum_{k=n_0}^{n-1} q^{n-k} \cdot V^b + V^b \\ &\subseteq \sum_{k=1}^{n_0-1} |a_k \dots a_{n_0-1}| \operatorname{conv}(V^b) + \frac{q}{1-q} \operatorname{conv}(V^b) + \operatorname{conv}(V^b) \\ &= \left(\sum_{k=1}^{n_0-1} |a_k \dots a_{n_0-1}| + \frac{1}{1-q} \right) \operatorname{conv}(V^b). \end{aligned}$$

Choosing

$$M := \sum_{k=1}^{n_0-1} |a_k \dots a_{n_0-1}| + \frac{1}{1-q}.$$

□

In Theorem 2.7 the sequence $(y_n)_{n \geq 0}$ is not necessary uniquely determined (see [14, Remark 2.2]).

Theorem 2.8. *Let \mathcal{X} be a Hausdorff topological vector space over the field \mathbb{K} , V a bounded subset of \mathcal{X} , $a \in \mathbb{K}$, $|a| < 1$, and $(b_n)_{n \geq 0}$ a sequence in \mathcal{X} . Then for every sequence $(x_n)_{n \geq 0}$ in \mathcal{X} satisfying (2.9) there exists a sequence $(y_n)_{n \geq 0}$ in \mathcal{X} fulfilling the recurrence (2.10) such that*

$$x_n - y_n \in \frac{1}{1-|a|} \cdot \operatorname{conv}(V^b) \quad (n \geq 0).$$

Proof. Let $c_n := x_{n+1} - ax_n - b_n$ ($n \geq 0$) and $(y_n)_{n \geq 0}$ be given by the recurrence (2.10) with $y_0 = x_0$. Then

$$x_n - y_n = \sum_{k=1}^n a^{n-k} c_{k-1} \quad (n \geq 1).$$

It follows that

$$\begin{aligned} x_n - y_n &\in \sum_{k=1}^n a^{n-k} V \subseteq \sum_{k=1}^n |a|^{n-k} V^b \\ &\subseteq \sum_{k=1}^n |a|^{n-k} \text{conv}(V^b) \\ &= \left(\sum_{k=1}^n |a|^{n-k} \right) \text{conv}(V^b) \\ &= \frac{1 - |a|^n}{1 - |a|} \text{conv}(V^b) \\ &\subseteq \frac{1}{1 - |a|} \text{conv}(V^b) \quad (n \geq 0). \end{aligned}$$

□

Corollary 2.9. *Suppose that \mathcal{X} is a normed space over \mathbb{K} , $\varepsilon > 0$, $|a| < 1$ and $(b_n)_{n \geq 0}$ is a sequence in \mathcal{X} . Then for every sequence $(x_n)_{n \geq 0}$ in \mathcal{X} satisfying*

$$\|x_{n+1} - ax_n - b_n\| \leq \varepsilon \quad (n \geq 0)$$

there exist a positive number M and a sequence $(y_n)_{n \geq 0}$ satisfying the recurrence

$$y_{n+1} = ay_n + b_n \quad (n \geq 0)$$

such that

$$\|x_n - y_n\| \leq \frac{1}{1 - |a|} \varepsilon \quad (n \geq 0).$$

Proof. Use Theorem 2.8 with $a_n = a$ and take V to be the closed ball of center 0 with radius ε . □

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